ASSET PRICING FOUNDATIONS OF INVESTMENT STRATEGIES

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3. MACRO-FINANCE ASSET PRICING MODELS

Road Map

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  - Recursive Preferences
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A Quick Overview

Macro-finance studies the relationship between asset prices (expected returns) and economic fluctuations

A Quick Overview

Macro-finance asset pricing models look for consumption (investment)-based aggregate variables that are **good proxies for aggregate marginal utility growth**, i.e., variables for which

$$M_{t+1} = \rho \frac{U'(C_{t+1})}{U'(C_t)}$$

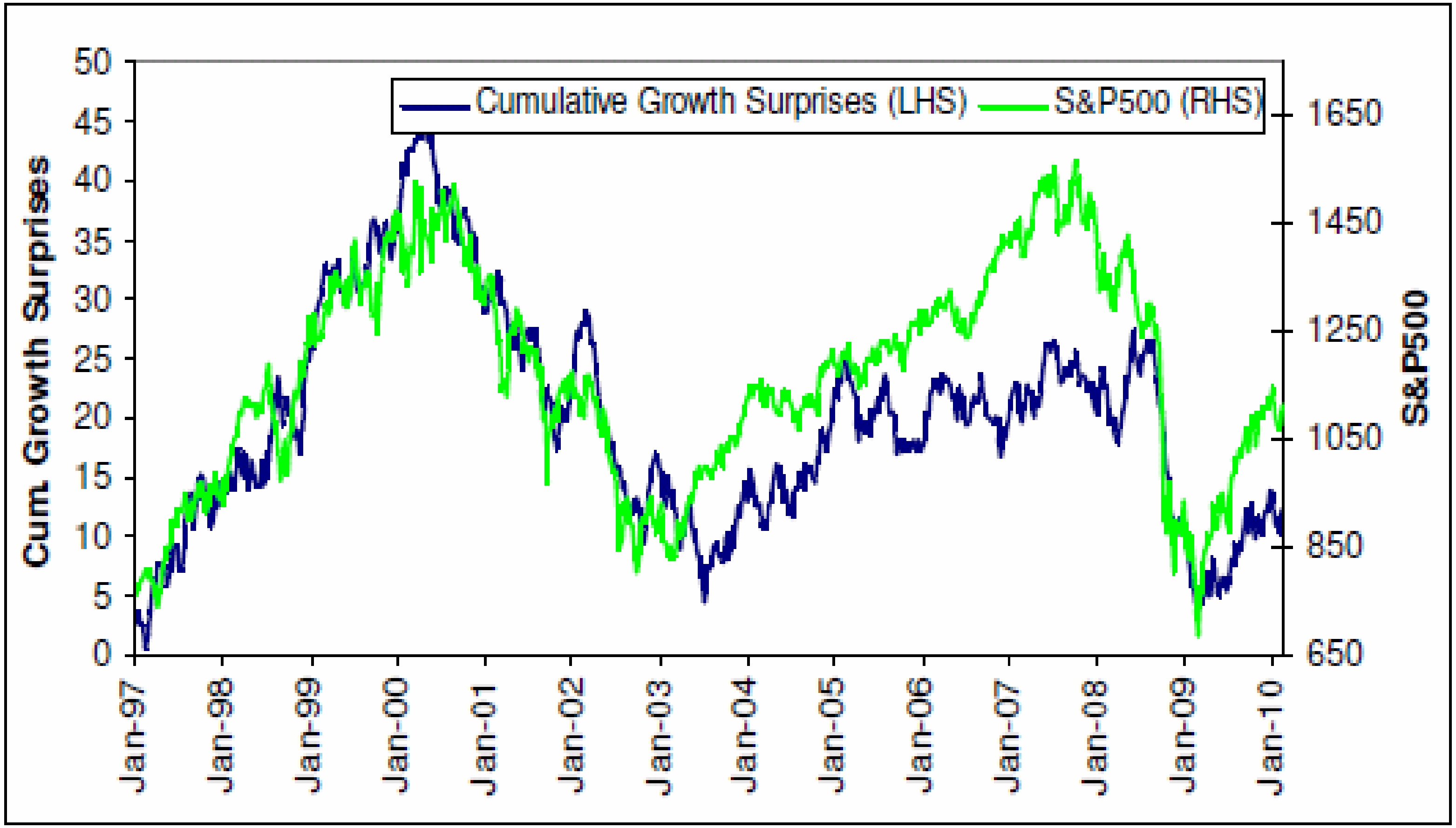
can be described in a sensible and economically interpretable approximation:

A good consumption-based index of "bad times"

The big question is, what should one use for preferences to generate enough volatility in the growth rate of marginal utilities (SDF)?

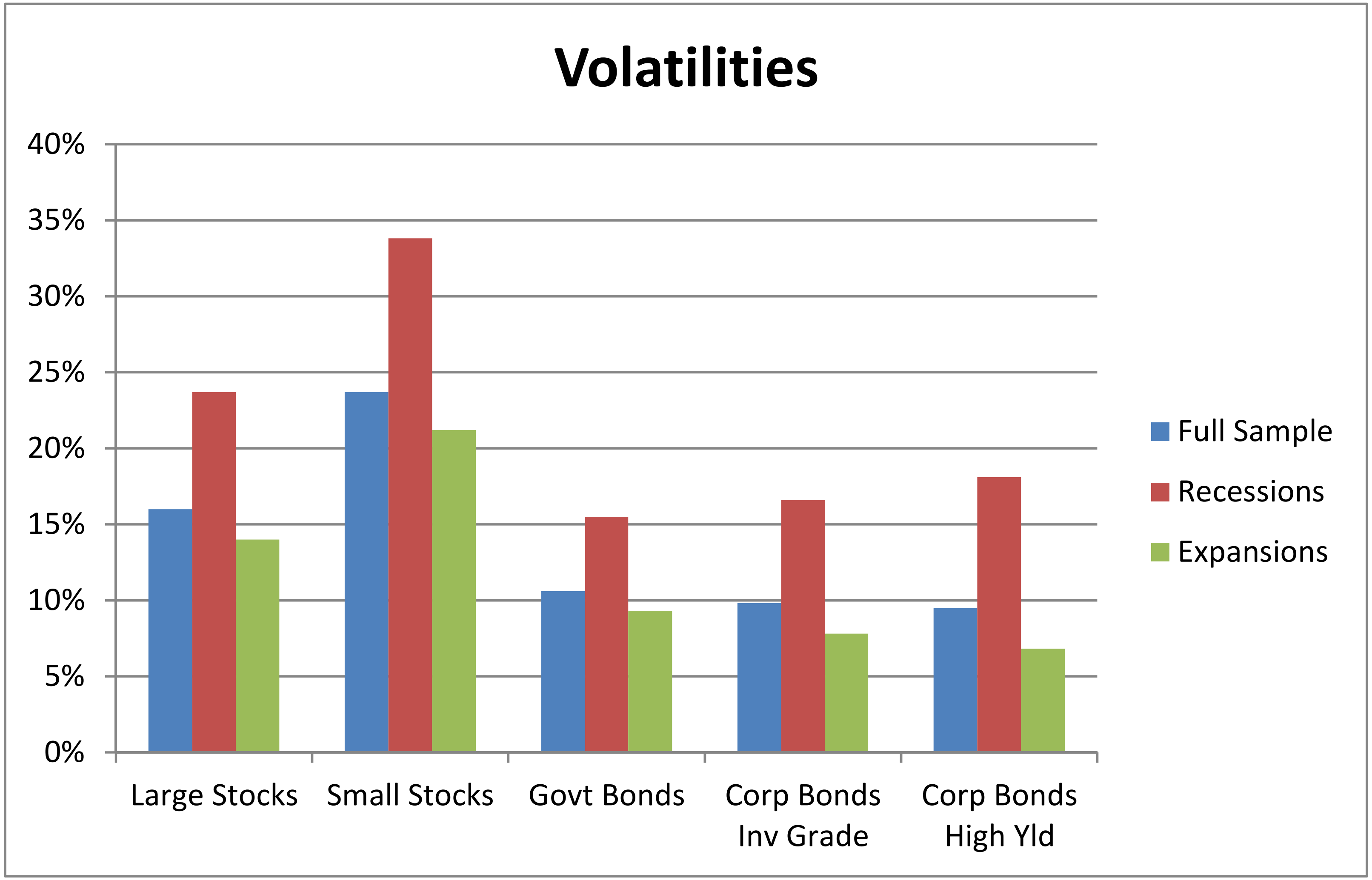
Macroeconomic Factors and the Consumption-Based Asset Pricing Model

US equities have closely tracked economic growth surprises well



Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Economic Growth



Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The SDF is the MRS between consumption over time and is given by

$$M_{t+1} = \rho \frac{U'(C_{t+1})}{U'(C_t)}$$

We now we give empirical content to the basic pricing equation by imposing a specific functional form to the utility function

Asset pricing often uses a **power utility function** of the form

$$U(C_t) = \begin{cases} \dfrac{C_t^{1-\gamma} - 1}{1-\gamma} \; ; \; \gamma > 0, \gamma \neq 1 \\[2ex] ln(C_t) \; ; \; \gamma = 1 \end{cases}$$

where

$$R(C) = -C_t \frac{U''(C_t)}{U'(C_t)} = \gamma$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Under power utility the SDF is therefore,

$$M_{t+1} = \rho \frac{U'(C_{t+1})}{U'(C_t)} = \rho \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

In this case the SDF is determined by ***aggregate consumption growth*****, the *****relative risk aversion***** coefficient, and the *****impatience***** parameter**

It should be noted that this SDF specification is, indeed, counter-cyclical

M is high (low) at the beginning of recessions (expansions)

The relative risk aversion coefficient *magnifies* the counter-cyclical behavior of the SDF

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The basic pricing equation (first order condition) under power utility is

$$E_t\left\lfloor M_{t+1}R_{jt+1}\right\rfloor = 1$$

$$E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}R_{jt+1}\right] = 1;\ 1, \dots, N$$

If we want to estimate the model, we have to estimate two parameters (impatience, relative risk aversion), while the data is aggregate consumption growth, and the *N* rates of returns of the available assets

For the risk-free rate,

$$E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]R_{ft+1} = 1$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The model is non-linear; to provide the intuitive and important implications of the model, we obtain the linear specification of the consumption-based asset pricing model

Linearizing the model assuming that asset prices and consumption are **log-normal variables**

Recall that if any random variable, lnY is normal, then Y is lognormal. Therefore,

$$\begin{cases} X \approx Normal \\ \Rightarrow ln\, e^{X} \approx Normal \\ \Rightarrow e^{X} \approx Lognormal \end{cases}$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Properties of log-normal random variables

$$If \;\; X \; is \; N \rightarrow e^{X} \;\; is \; lnN, then$$

$$E\left(e^{X}\right) = e^{E(X) + \frac{1}{2} Var(X)}$$

Taking logarithms, we obtain

$$ln\, E\left(e^{X}\right) = E(X) + \frac{1}{2} Var(X) = E\left[ln\left(e^{X}\right)\right] + \frac{1}{2} Var\left[ln\left(e^{X}\right)\right]$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

We linearize first the expression for the risk-free rate taking logarithms in both sides of the equation to get

$$ln\, R_{ft+1} = -ln\, E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]$$

Then, we assume lognormality and apply the previous properties:

$$ln\, R_{ft+1} = -E_t\left[ln\left(\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right] - \frac{1}{2}Var_t\left[ln\left(\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right]$$

$$= -ln\,\rho + \gamma E_t\left[ln\left(\frac{C_{t+1}}{C_t}\right)\right] - \frac{\gamma^2}{2}Var_t\left(ln\left(\frac{C_{t+1}}{C_t}\right)\right)$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Let lower cases denote the logarithm of any variable,

$$E_t(\Delta c_{t+1}) \equiv E_t\left[ln\left(\frac{C_{t+1}}{C_t}\right)\right]$$

$$\sigma_C^2 \equiv Var_t\left[ln\left(\frac{C_{t+1}}{C_t}\right)\right]$$

Therefore, we can conclude that the riskless rate is determined by the following expression

$$r_{ft+1} = -ln\,\rho + \gamma\, E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2}\sigma_C^2$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The risk-free interest rates under power utility is given by

$$r_{ft+1} = -ln\,\rho + \gamma\, E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2}\sigma_C^2$$

Real interest rates are **high when impatience is high**,

$$impatience \equiv \delta = -ln(\rho) \Rightarrow \rho = e^{-\delta}$$

$$high\, impatience \Leftrightarrow low\ \rho$$

$$\rho < 1 \Rightarrow ln\,\rho < 0 \Rightarrow (-)(-) \Rightarrow (+)$$

and when **future consumption growth is expected to be high** (higher risk aversion makes interest rates more sensitive to consumption growth), and when **consumption becomes more volatile** people want to save more driving down interest rates (precautionary savings)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

We can now define the ***elasticity of intertemporal substitution*** as

$$\eta \equiv \frac{\partial \, ln\left(C_{t+1} / C_t\right)}{\partial \, ln\left(R_{ft+1}\right)} = \frac{\partial \Delta \, c_{t+1}}{\partial \, r_{ft+1}} = \frac{1}{\gamma}$$

Hence, ***with power utility, the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion***

That is, the single parameter γ determines both risk aversion (across states) and the rate of intertemporal susbstitution (risk aversion across time)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

We now linearize the expression for the risky asset

$$E_t\left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{jt+1}\right]=1$$

By repeating the same steps as the ones made for the risk-free rate,

$$E_t\left[r_{jt+1}\right]=-\ln\rho+\gamma\, E_t[\Delta c_{t+1}]-\frac{1}{2}\left(\gamma^2\sigma_C^2+\sigma_j^2-2\,\gamma\,\sigma_{jC}\right)$$

where $\sigma_j^2 \equiv Var_t\left(r_{jt+1}\right)$ and $\sigma_{jc} \equiv Cov_t\left(r_{jt+1},\Delta c_{t+1}\right)$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Subtracting the risk-free rate from the expression for the excess return of any asset *j*, we get

$$E_t\left[r_{jt+1}\right]-r_{ft+1}+\underbrace{\frac{1}{2}\sigma_j^2}_{\substack{\text{Jensen's} \\ \text{inequality term}}}=\gamma\,\sigma_{jc}$$

Consumption-based asset pricing model:

The expected excess return on any asset is linear and positively related to the covariance of aggregate consumption growth (the factor) with the return of the asset where the slope is the relative risk aversion coefficient

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

$$E_t\left[R_{jt+1}\right] - R_{ft+1} \cong \gamma\, \sigma_{jc}$$

- An asset whose return covaries positively with aggregate consumption growth **makes consumption growth more volatile** and must promise higher expected excess returns

- In other words, in this model, **bad times are defined by low consumption growth**: if the asset's payoffs are high (low) when consumption is high (low), the asset is risky and the investors require a high expected return on the asset

- Moreover, the expected excess return will be higher the **more risk averse** the representative investor is

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

$$E_t\lfloor R_{jt+1}\rfloor - R_{ft+1} \cong \gamma\,\sigma_{jc}$$

- **Equity carries a premium because it covaries positively with bad consumption outcomes**. When consumption is low (marginal utility is high), equity returns are low, and thus the representative agent must be compensated for bearing equity risk in the form of the equity premium

- Other factors matter only so far as they affect consumption

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Power utility has two main inconvenient theoretical characteristics

- **Constant relative risk aversion coefficient over time and across states**

- This implies that the agent is willing to pay the same price to avoid risk independently of the expectations about the state of the economy

- However, the empirical evidence suggests that investors are more risk averse during recessions

- This implies a counter-cyclical risk aversion (higher at business-cycle troughs than at peaks)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

- **The elasticity of intertemporal substitution is exactly the inverse of the relative risk aversion coefficient**

- This time-separable utility function forces an inverse relationship between two parameters that should not be necessarily connected (aversion to consumption volatility across states and aversion to consumption volatility over time)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The Equity Premium "Puzzle"

Variables	Mean	Standard deviation	Covariance with consumption growth
Consumption growth	1.72%	3.28%	0.0011
Market return	6.01%	16.74%	0.0027
Risk-free rate	1.83%	5.44%	-0.0002

Annual U.S. data from **1889 to 1994** (Campbell and MacKinley)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Given that the market average return has been 6% and the average risk-free 1.8%, the market risk premium during these 100 years has been 4.2%

Given the average statistics for per capita, non-durable consumption growth, what should be the risk aversion coefficient for the model to explain the average risk premium?

$$E(r_j) - r_f = -\frac{\sigma_j^2}{2} + \gamma\,\sigma_{jc}$$

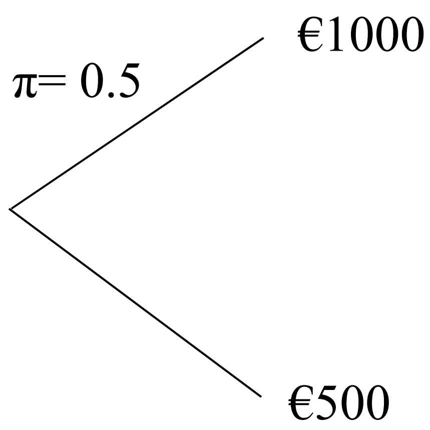
$$0.0418 = -0.1674^2 / 2 + \gamma\, 0.0027$$

$$\Rightarrow \gamma = 20.67$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

What risk aversion levels do investors have?

How much would you pay for the following bet? [Calibrated with CRRA utility]



γ	Amount you would pay
0	750
0.5	729
1	707
2	667
3	632
4	606
5	586
10	540
15	525
20	519
50	507

You will get 500 for sure, but you have the possibility of winning 1,000

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

An individual not willing to risk anything (you get €500 for sure) is infinitely risk averse

An individual willing to pay the fair value of the gamble, which is €750, is risk neutral and has risk aversion of $\gamma = 0$

An individual willing to risk more than the fair value is risk seeking

Most individuals have risk aversion between 1 and 10; it is very rare to have a risk aversion greater than 10. That is, most people would be willing to pay between €540 and €707 to enter this lottery

Based on actual financial decisions in an online person-to-person platform (Columbia University): around 3

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

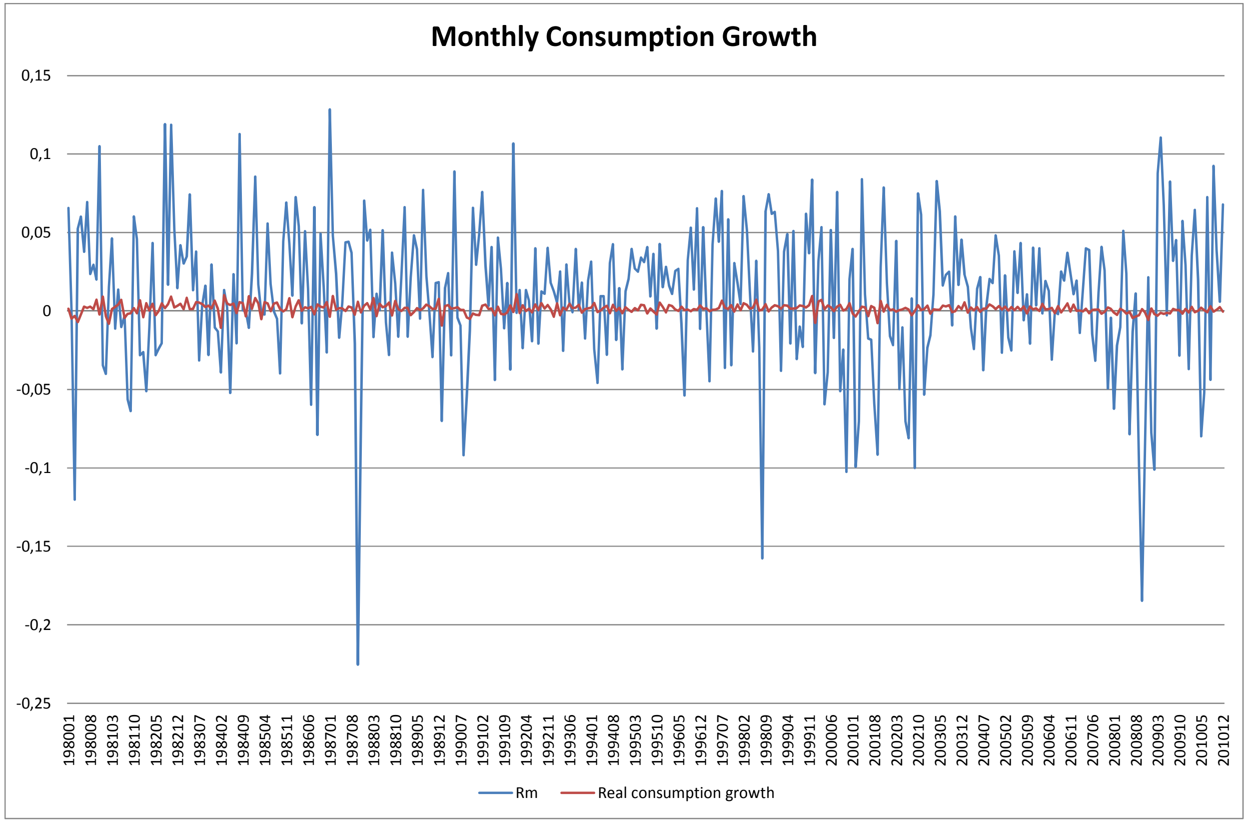
The postwar (quarterly data from 1947 to 2010) mean-value of the market risk premium is about 7% over the T-bill, with a volatility of about 16.5%: to match the equity premium, the model needs a risk aversion higher than 100

Why doesn't consumption explain equity premiums unless we impose a high (even huge) risk aversion coefficient?

- ► Equity volatility is high (~15-20%) while consumption volatility is low (~2-3%)

- ► Equity returns are lowly correlated with consumption growth (~10-15%)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model



Consumption growth seems to be too smooth to be able to explain market returns: their covariance is too small to explain average excess returns; we need a very high risk aversion coefficient!

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

It has been traditional to use (as reasonable levels) risk aversion numbers of 1 to 10, but perhaps this is tradition, not fact. What is wrong with $\gamma = 21$, 50 or higher?

We take the expression for the risk-free rate and analyze the consequences of these levels of risk aversion coefficients

$$r_{ft+1} = -\ln\rho + \gamma\, E_t(\Delta c_{t+1}) - \frac{\gamma^2}{2}\sigma_C^2$$

We assume a reasonable subjective discount factor for preferences ($\rho = 0.98203$):

$$r_f = -\ln 0.98203 + 20.67 \times 0.0172 - \frac{1}{2} \times 20.67^2 \times 0.033^2$$

$\Rightarrow r_f = 0.1410$ → **Much higher than the historical rate!!!**

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

To understand this contradiction (*huge risk aversion to explain the market risk premium but then, extremely high interest rates*) we should realize that this consumption pricing model with time-separable power utility forces an inverse relationship between risk aversion and the elasticity of intertemporal substitution

When risk aversion is high the model implies that the elasticity of intertemporal substitution is low and viceversa

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Risk aversion coefficients of 21 to 100 implies that investors are essentially unwilling to substitute (expected) consumption over time (the elasticity would be between 0.05 and 0.01), so only a very high interest rate, and huge interest rate variation would force them to make the relatively small variations in consumption growth that we do see

It seems clear that asset pricing needs a utility function in which people are reasonably risk averse across states and still willing to substitute consumption over time (they should be independent parameters)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

Another useful way to discuss the equity premium puzzle is recalling the HJ-volatility bound

$$\left|\frac{E\left(R_j^e\right)}{\sigma\left(R_j^e\right)}\right| \leq \frac{\sigma(M)}{E(M)}$$

Under power utility,

$$\left|\frac{E\left(R_j^e\right)}{\sigma\left(R_j^e\right)}\right| \leq \frac{\sigma\left[\left(C_{t+1}/C_t\right)^{-\gamma}\right]}{E\left[\left(C_{t+1}/C_t\right)^{-\gamma}\right]}$$

Assuming that consumption growth is lognormal,

$$\left|\frac{E\left(R_j^e\right)}{\sigma\left(R_j^e\right)}\right| \leq \gamma\,\sigma_c$$

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

From 1926 to 2015, the Sharpe ratio for the market is 0.42

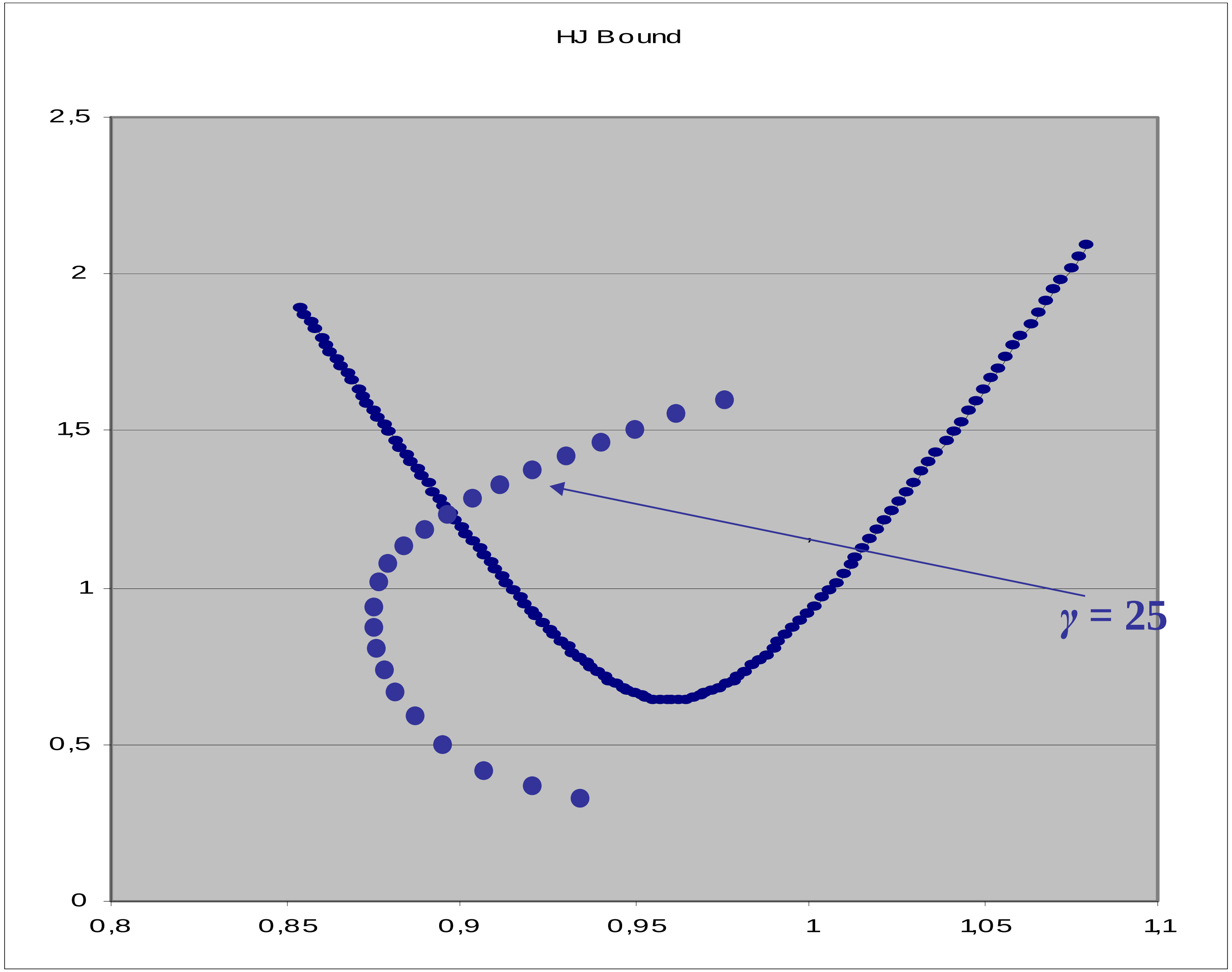
$$\gamma \geq \frac{\textit{Sharpe Ratio}}{\sigma_c} = \frac{0.42}{0.025} = 17$$

If the stock market portfolio were less than efficient (as it is the case), so a strict inequality holds, the magnitude of the risk aversion coefficient would need to be even higher

Again, we have the inconsistency between theory and empirical evidence (equity premium puzzle)

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

1958-2006 25 Fama-French Portfolios for Alternative Risk Aversion Coefficients



Macroeconomic Factors and the Consumption-Based Asset Pricing Model

The basic consumption-based asset pricing model with power utility seems unable to explain financial market data

So what?

• Understanding why the equity premium exists is important for understanding whether equities should have high returns in the future

• Recommending an optimal allocation to equities necessitates understanding why equities deliver high returns over the long run, and whether we can stomach years (even decades!) of sub-performance

Macroeconomic Factors and the Consumption-Based Asset Pricing Model

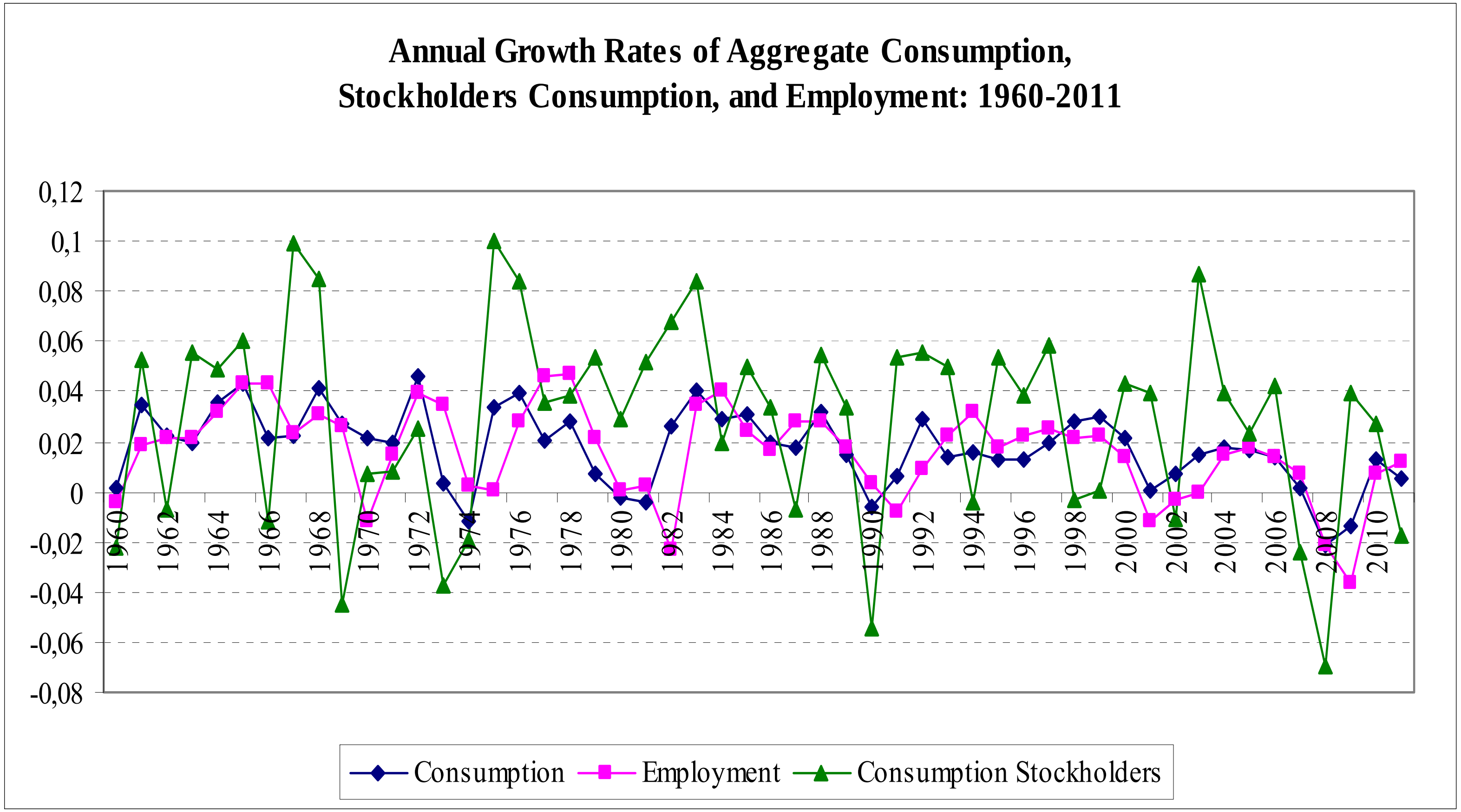
Is the equity premium puzzle a puzzle?

• What about a different utility function? What if risk aversion does not have to be linked to then elasticity of intertemporal substitution? What if habits are important and preferences are not separable over time, and across states?

• What about non-separable arguments in the utility function rather only consumption? *U(C, Z)* where *C* and *Z* are not separable

• What is not everybody does not hold stocks? Stockholders consumption?

Macroeconomic Factors and the Consumption-Based Asset Pricing Model



Consumption-Based Asset Pricing Models with Non-Separability

Macroeconomic-based asset pricing models require a **volatile stochastic discount factor**

How do we accomplish this? The basic SDF for power utility is

$$M_{t+1} = \rho \frac{U'(C_{t+1})}{U'(C_t)} = \rho \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$$

We need an extra variable that varies over time with recessions!

$$M_{t+1} = \rho \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} Y_{t+1}$$

The new variable Y_{t+1} does most of the work and account for essentially all the entire market risk premium

Consumption-Based Asset Pricing Models with Non-Separability

• To make any progress in pricing financial assets, any extra state-variable needs to be sensible enough to shifts in marginal utility

• It should contain as few extra assumptions as possible

• It should generate the extra state variable **endogenously**

• **The extra variable must be a recession-related state variable**

• **The tendency for assets to fall when Y_{t+1} is bad drives risk premiums, and changes in the conditional density of Y_{t+1} drive time-varying risk premiums**

Consumption-Based Asset Pricing Models with Non-Separability

- The macro-finance models all describe a market with a time-varying ability to bear risk

- The source of that time-varying risk-bearing ability is the primary difference among available models

Consumption-Based Asset Pricing Models with Non-Separability

- Habit-based models (Campbell and Cochrane, JPE 1999)
- Recursive utility (Epstein and Zin, Econometrica 1989)
- Long run risks (Bansal and Yaron 2004, Hansen, Heaton, and Li, JPE 2008)
- Idiosyncratic risk (Constantinides and Duffie, JPE 1995)
- Heterogeneous preferences (Garleanu and Panageas, JPE 2015)
- Leverage/Institutional Finance (Brunnermeier, JEP 2009; Adrian, Etula, and Muir, JF 2014)
- Rare disasters (Reitz, JME 1988; Barro, QJE 2006)

Consumption-Based Asset Pricing Models with Non-Separability

• We may give the same intuition working directly with the utility function rather than with the SDF. We just have to add arguments to the utility function in a **NON-SEPARABLE FASHION**

In a utility context, we add non-separable arguments to the utility function *U(C, Z)*, so

$$E_t\left(R_{jt+1}\right) - R_{ft+1} \cong \gamma \, Cov_t\left(R_{jt+1}, \Delta C_{t+1}\right) + \lambda_Z^C \, Cov_t\left(R_{jt+1}, \Delta Z_{t+1}\right)$$

where

$$\gamma \equiv -\frac{CU_{CC}}{U_C} \qquad \lambda_Z^C \equiv \frac{ZU_{CZ}}{U_C}$$

The extra utility function arguments must enter **non-separably**; otherwise

$$U(C, Z) = f(C) + g(Z)$$

and then, $U_{CZ} = 0$

Consumption-Based Asset Pricing Models with Non-Separability

• Using this framework, we now go back to asset pricing models, and discuss **two types of non-separability in the utility function or, alternatively, we explore two candidates for the extra variable *Y* in the SDF**

- ***Recursive preferences*** (as a very important special case, we obtain the ICAPM, and the so called long-run risk models; it is also the key model in the dynamic capital structure models of Corporate Finance)

- ***External habit preferences model with time-varying risk aversion***

Consumption-Based Asset Pricing Models with Non-Separability

Recursive Preferences

• Preferences are specified by assuming that utility is given moment of time and state depends on consumption at that moment and state and also on the utility obtained throughout consumption in other moments and states in the future:

$$U_t\left(C_t, E_t\left(U_{t+1}\right)\right)$$

Hence, marginal utility on a given moment and state affects what happens in other moments and states: ***utility is temporal and state dependent***

Consumption-Based Asset Pricing Models with Non-Separability

• Epstein and Zin (1989) and Weil (1989) build on the approach of ***recursive preferences*** proposed by Kreps and Porteus where ***risk aversion and the elasticity of intertemporal substitution are not directly linked***

Let η be the elasticity of intertemporal substitution, and $\kappa \equiv 1/\eta$

The utility function is

$$U_t = f\left(C_t, E_t\left(U_{t+1}\right)\right) = \left[\left(1-\rho\right)C_t^{1-\kappa} + \rho\left(E_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1-\kappa}{1-\gamma}}\right]^{\frac{1}{1-\kappa}}$$

Consumption-Based Asset Pricing Models with Non-Separability

• Under these preferences, $f(\ldots,\ldots)$ is the so called "aggregator function" which aggregates current consumption with future (expected) utility **obtaining non-separability**

• If the aggregator function is linear in its second argument, we obtain the state-time-separable expected utility representation

• Hence, these are a very general family of preferences in which the well known power utility is a special case for $\gamma = \kappa$

Consumption-Based Asset Pricing Models with Non-Separability

• It captures temporal dependency (and state) without recurring to past consumption (this will be the case for habit preferences), but incorporating future consumption throughout its expected utility

• Moreover, **it breaks the link** between γ and η

• When $\gamma > \kappa$ the agent shows preference for early resolution of uncertainty; when $\gamma < \kappa$ preference for late resolution, but it also includes the for which the agent is indifferent $\gamma = \kappa$ (traditional expected utility case)

Consumption-Based Asset Pricing Models with Non-Separability

- The SDF, or growth in marginal utility is

$$M_{t+1} = \rho\left(\frac{C_{t+1}}{C_t}\right)^{-\kappa}\left(\frac{U_{t+1}}{\left[E_t\left(U_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{\kappa-\gamma}$$

$$= \rho\left(\frac{C_{t+1}}{C_t}\right)^{-\kappa}\left(Y_{t+1}\right)^{\kappa-\gamma}$$

Therefore, the innovation in the utility index takes the role of the new variable *Y*

Consumption-Based Asset Pricing Models with Non-Separability

We simplify notation, by denoting $\theta \equiv \frac{1-\gamma}{1-\kappa}$

Then, the intertemporal consumption-investment problem of the representative agent is maximize expected utility,

$$MaxU_t \Rightarrow Max\left[(1-\rho)C_t^{\frac{1-\gamma}{\theta}} + \rho\left(E_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}$$

Subject to the intertemporal budget constraint,

$$W_{t+1} = (W_t - C_t)R_{mt+1}$$

Consumption-Based Asset Pricing Models with Non-Separability

After a complex mathematical manipulation (see Cochrane, 2008 for a simple but rigorous proof), it can be shown that under certain assumptions, the stochastic discount factor is given by

$$M_{t+1} = \left[\rho\left(\frac{C_{t+1}}{C_t}\right)^{-\kappa}\right]^{\theta} R_{mt+1}^{\theta-1}$$

When utility only depends on consumption at any point of time, the only source of risk is covariance with consumption growth

► **Now, utility also depends on the utility expected in the future, which is a function of the returns on new investments (market return)**

► Then, we have a second term in the stochastic discount factor and a second covariance term with the return on the market

Consumption-Based Asset Pricing Models with Non-Separability

HABIT

A hedge-fund manager's wife in a cocktail party: *"I'd sooner die than fly commercial again"*

Consumption-Based Asset Pricing Models with Non-Separability

Habit Preferences (Campbell and Cochrane, JPE 1999)

Time-varying risk aversion is the key concept

$$E_t\left(R^e_{jt+1}\right)=\gamma\, Cov_t\left(R_{jt+1},\Delta c_{t+1}\right)$$

but we may also have

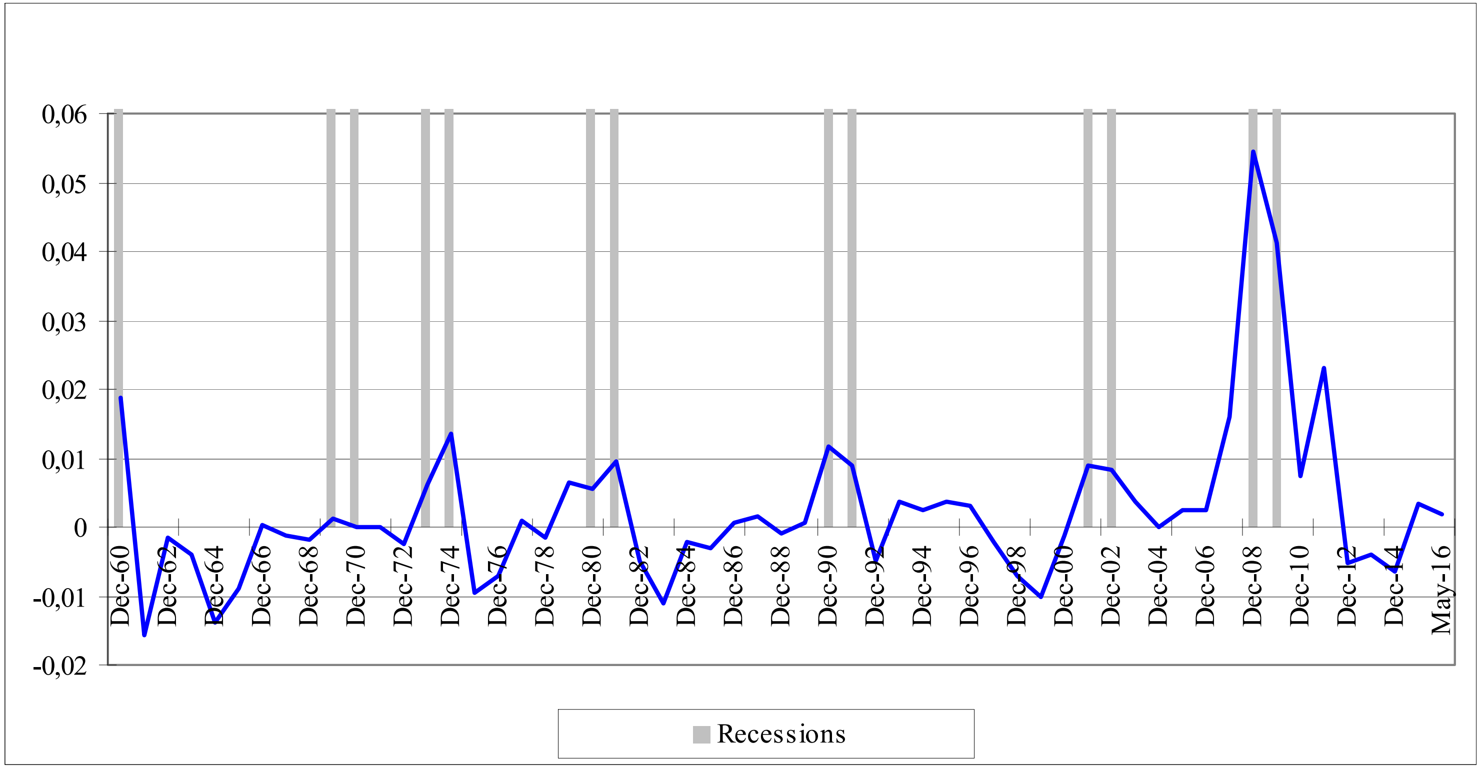
$$E_t\left(R^e_{jt+1}\right)=\gamma_t\, Cov\left(R_{jt+1},\Delta c_{t+1}\right)$$

► Now, expected returns are time-varying and counter-cyclical

► It makes sense to expect time-varying and counter cyclical risk aversion to mach expected returns

Consumption-Based Asset Pricing Models with Non-Separability

Yearly Changes inTime-Varying Risk Aversion 1960-2016 (May)



Consumption-Based Asset Pricing Models with Non-Separability

How do we get time-varying risk aversion?

We introduce **habit**, X, in the utility function, where habit and consumption are non-separable:

$$U_t = U(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

The SDF is given by

$$M_{t+1} = \rho\left(\frac{S_{t+1}}{S_t}\frac{C_{t+1}}{C_t}\right)^{-\gamma} = \rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\underbrace{\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}}_{Y_{t+1}}$$

where,

$$S_t = \frac{C_t - X_t}{C_t}$$

is a state variable known as **"surplus consumption ratio"** that allows to capture dependencies among states of nature and time

Consumption-Based Asset Pricing Models with Non-Separability

How do we get time-varying risk aversion?

$$\frac{\partial U}{\partial C_t} = S_t^{-\gamma} C_t^{-\gamma}$$

$$\frac{\partial^2 U}{\partial C_t^2} = -\gamma\, S_t^{-\gamma-1} C_t^{-\gamma-1}$$

$$RRA(S_t) = -C_t \frac{\partial^2 U}{\partial C_t^2} \Big/ \frac{\partial U}{\partial C_t} = \frac{\gamma}{S_t}$$

► Time-varying risk aversion is counter-cyclical exacting replicating the expected risk premium of risky assets

► When consumption surplus, *S*, goes down, and the economy enters into a recession, risk aversion increases (γ is the curvature parameter that provides a lower bound on the time-varying risk

Consumption-Based Asset Pricing Models with Non-Separability

The external (keeping up with the Joneses-aggregate consumption) habit specification presents time-varying counter-cyclical risk aversion

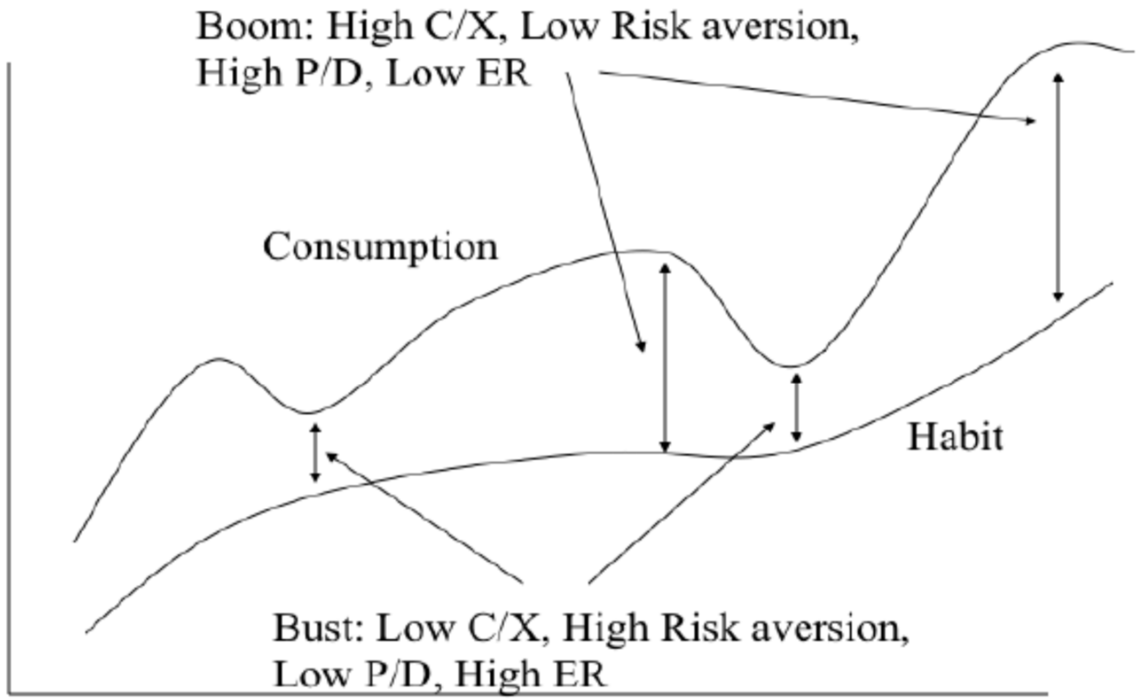
• Risk aversion should depend on **deviations** of consumption relative to its tendency (and not relative to its level), and this tendency is captured through external habit formation

• Hence, in their model, **risk aversion changes with the state of the economy in a counter-cyclical fashion exactly replicating the expected risk premia of risky assets**

Consumption-Based Asset Pricing Models with Non-Separability

The model makes the habit slow-moving

$$X_t \approx k \sum^{\infty} \phi^{\tau} C_{t-\tau}; \;\; X_t = \phi X_{t-1} + kC_t$$



$$E_t\left(R_{jt+1}\right) - R_{ft+1} \approx \gamma_t \, \sigma_{jc}$$

The model delivers a time-varying recession-driven risk premium

Consumption-Based Asset Pricing Models with Non-Separability

The model delivers a slowly varying interest rate

$$R_f \approx \rho + \gamma \left(\frac{X}{C - X} \right) E\left(\frac{dC}{C} \right) - \frac{1}{2}\gamma(\gamma + 1)\left(\frac{C}{C - X} \right)^2 \sigma_c^2$$

• The real interest rate is

subjective discount factor plus the elasticity of intertemporal substitution times expected consumption growth MINUS risk aversion squared times the variance of consumption growth

• In bad times, consumers want to borrow against future good times by ***intertemporal substitution***, but they want to save against the possibility of future risk by ***precautionary savings***; the model **offsets** these forces to produce a constant or slowly varying interest rates

Consumption-Based Asset Pricing Models with Non-Separability

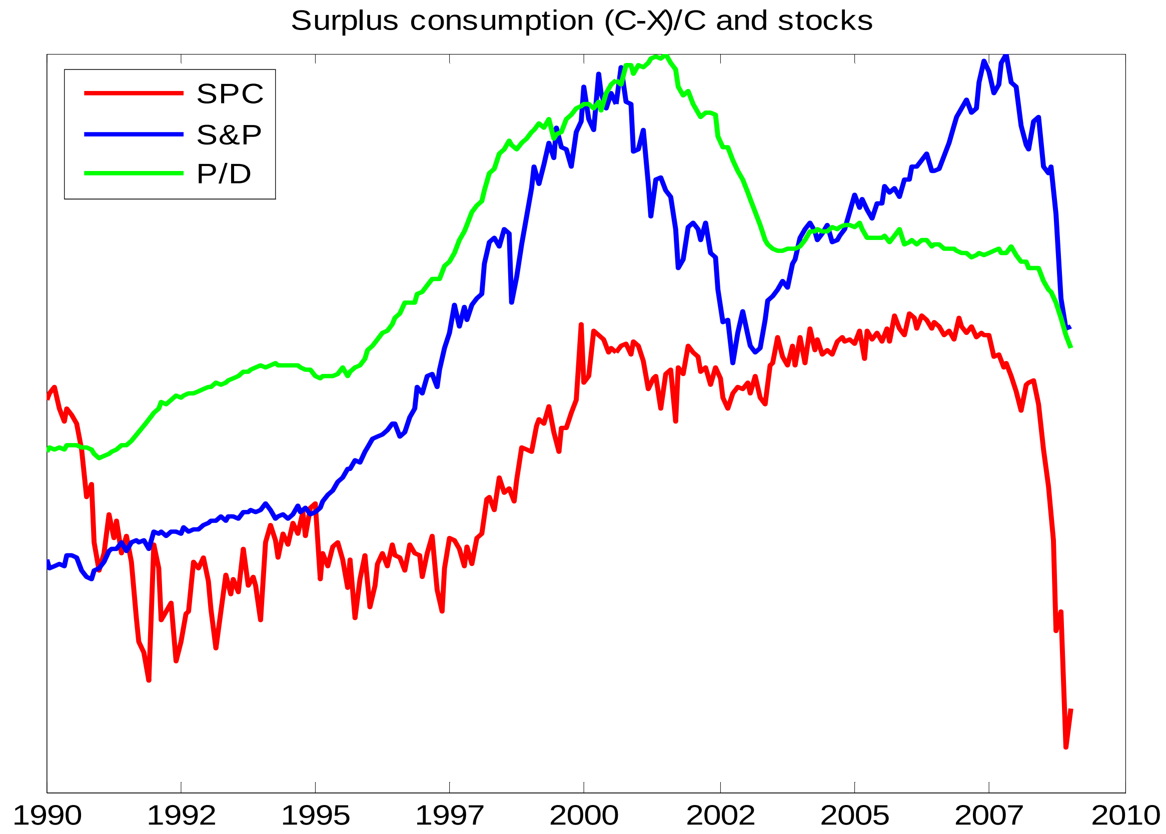
- The linear version of the model is given by

$$E_t\left(r_{jt+1}\right)-r_{ft+1}=-\frac{\sigma_j^2}{2}+\gamma\left(\lambda\left(s_t\right)+1\right)\sigma_{jc}$$

- As in the basic consumption pricing model, risk is given by the covariance between returns and consumption growth

- However, it is now the case that ***the price of risk is allowed to change over time with the state of the economy*** since the sensitivity parameter (how habit changes with consumption shocks) $\lambda(s_t)$, changes with the surplus consumption ratio

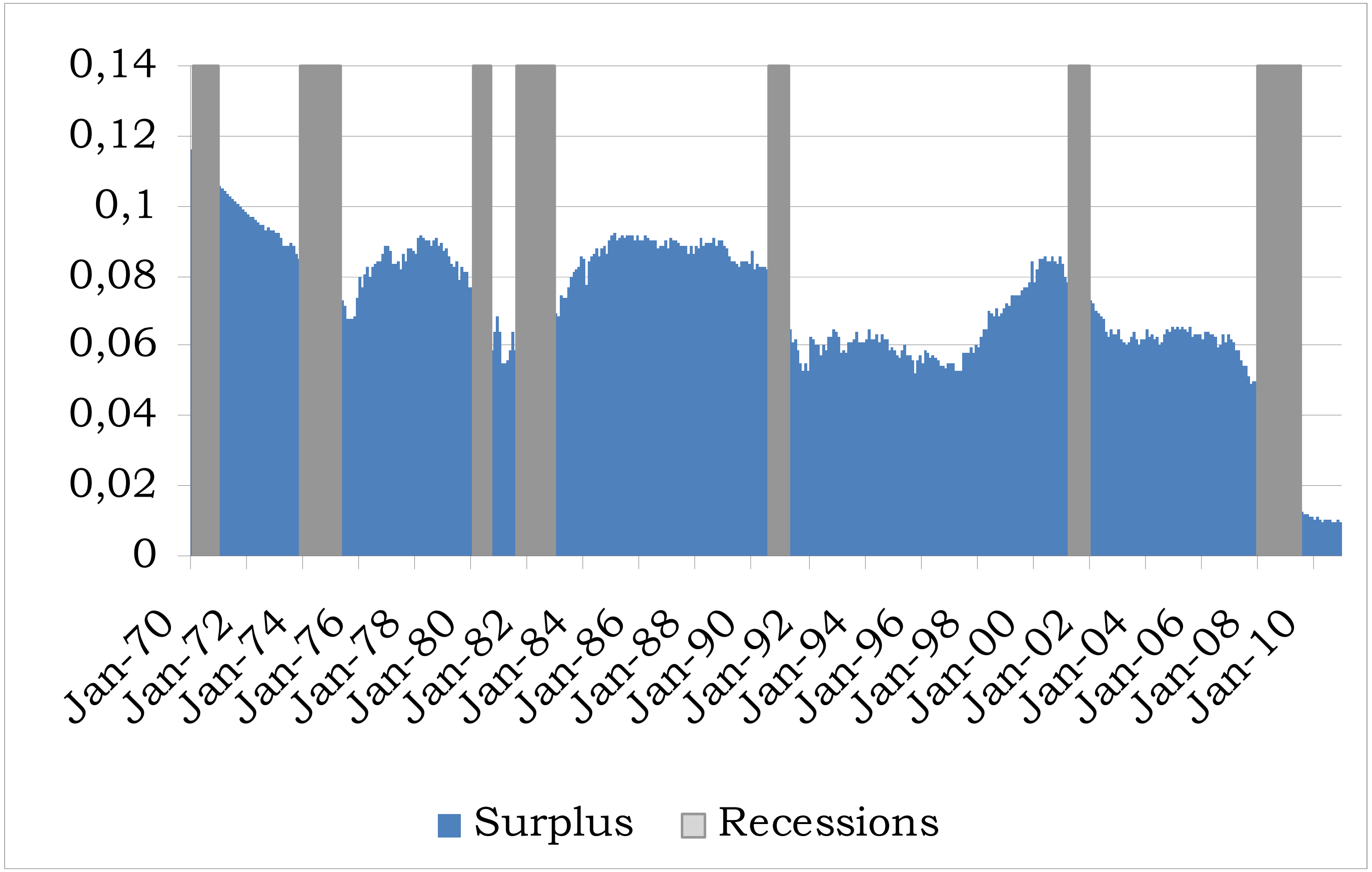
Consumption-Based Asset Pricing Models with Non-Separability



The model says that the *P/D* ratio should track the surplus consumption ratio; modern finance theory clearly accommodate financial crisis

Consumption-Based Asset Pricing Models with Non-Separability

Surplus Consumption Ratio 1970-2010



Consumption-Based Asset Pricing Models with Non-Separability

Going back to the recession-related state variable Y_{t+1}

$$M_{t+1} = \rho\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} Y_{t+1}$$

We can write the SDF as

$$M_{t+1} = m_{t+1} Y_{t+1}$$

where m_{t+1} is the SDF associated with a given model, and Y_{t+1} is the unobservable component

Consumption-Based Asset Pricing Models with Non-Separability

The Information Stochastic Discount Factor (I-SDF)

- Ghosh, Julliard, and Taylor (RFS, 2016) employ a model-free relative entropy minimization approach to estimate an (out-of-sample) SDF (M_{t+1}) that prices a given cross-section

- It is a non-parametric maximum likelihood estimate of the SDF; the most likely SDF and, therefore, the most likely one-factor pricing model for the cross-section used for its construction

- The relative entropy captures the divergence between the risk neutral probability Q and the physical probability P: this divergence measures the additional information content of Q relative to P

Consumption-Based Asset Pricing Models with Non-Separability

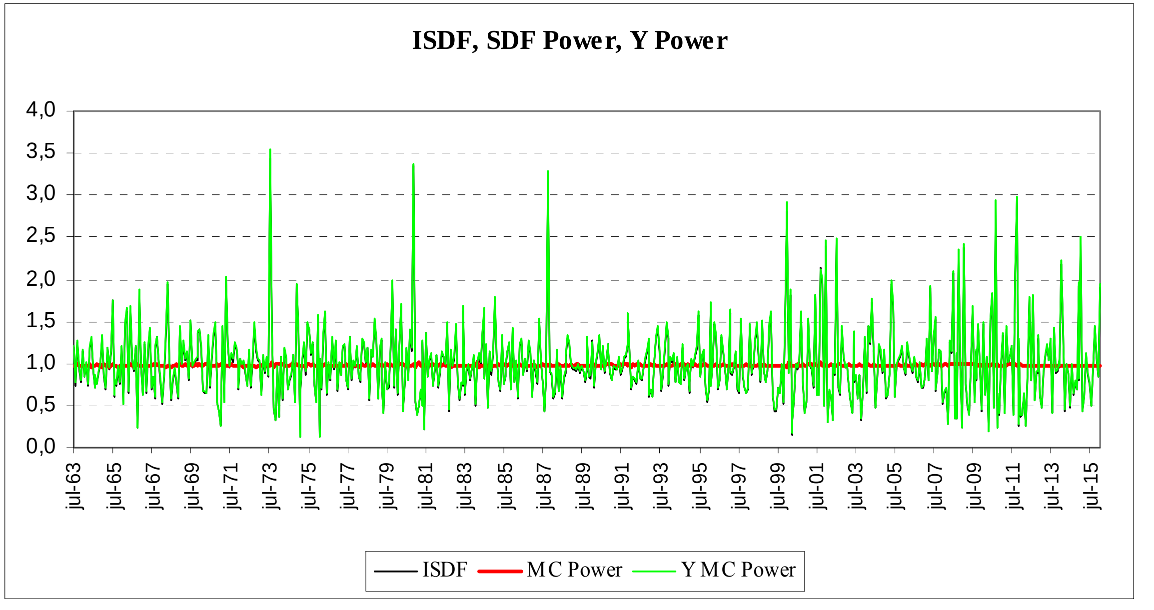
The Information Stochastic Discount Factor (I-SDF)

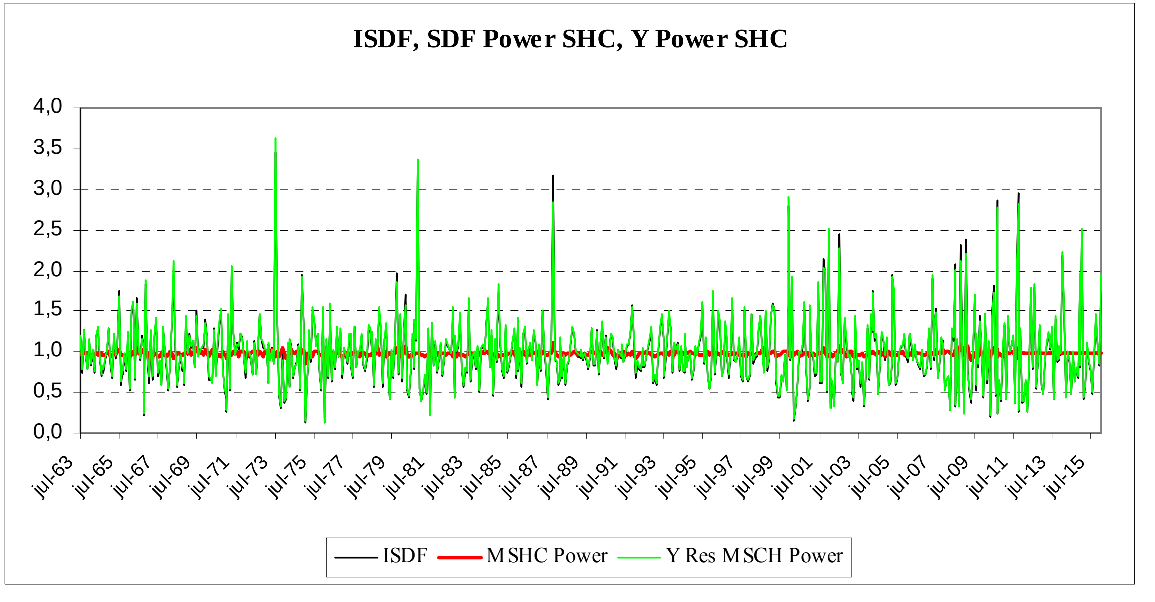
• Then, given the estimate of M_{t+1}, and the SDF, m_{t+1}, associated with a given model, we can extract Y_{t+1} and check what is missing from any asset pricing model

• I estimate the out-of-sample Information SDF, M_{t+1}, using the 25 FF portfolios by size and book-to-market

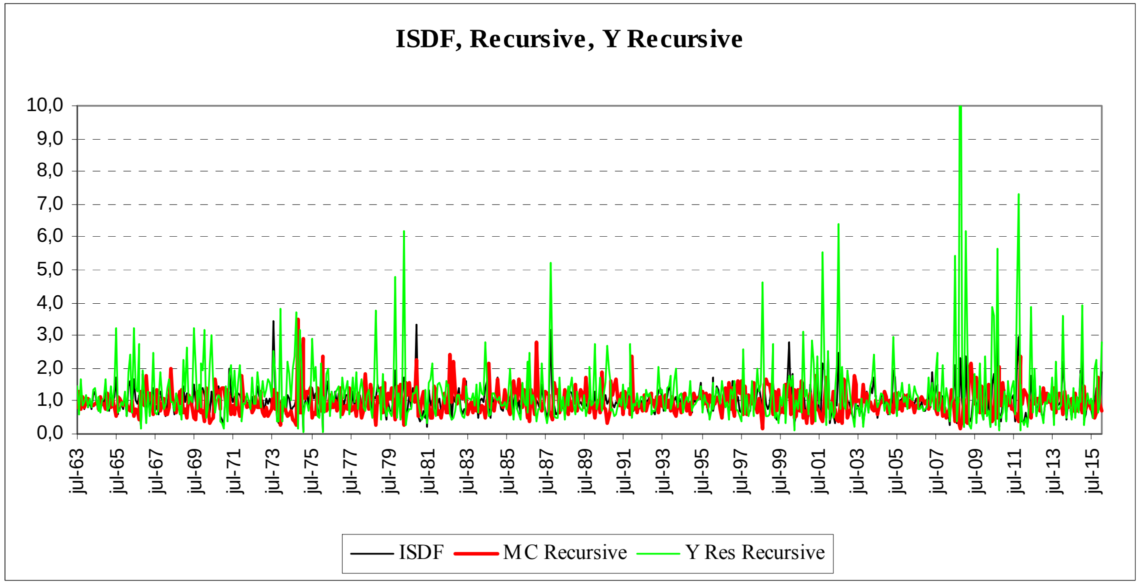
• In the first step, I estimate the Lagrange multipliers from the minimization restriction (θ_T) using monthly data from May 1933 to June 1963. I keep these estimates constant through the following year from July 1963 to June 1964, and calculate the monthly I-SDF during those 12 months. Then, I repeat the procedure with yearly rebalancing

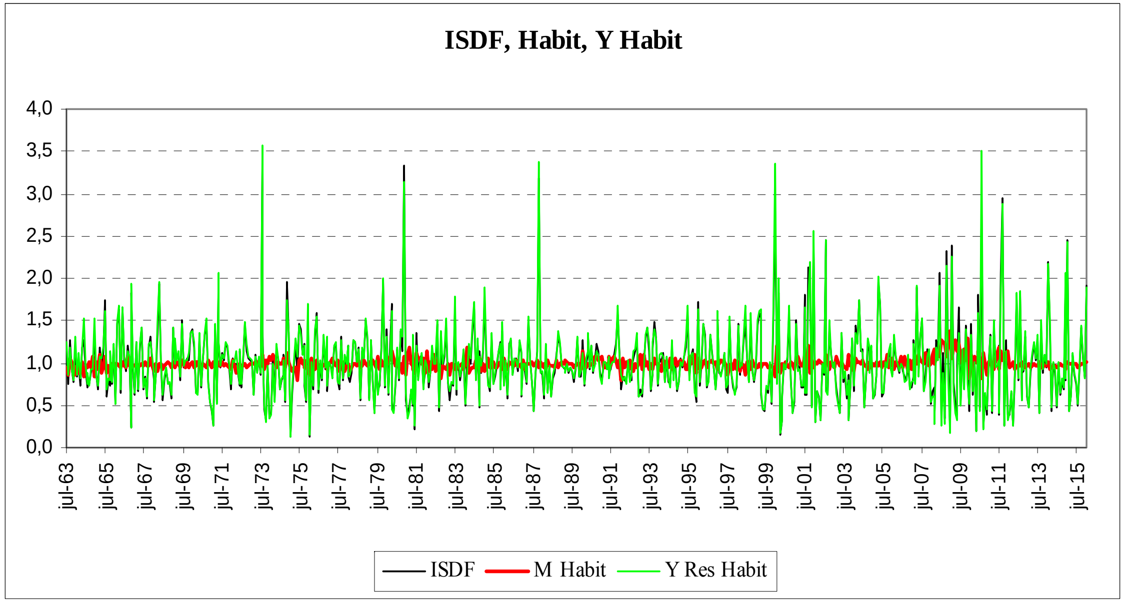
Consumption-Based Asset Pricing Models with Non-Separability



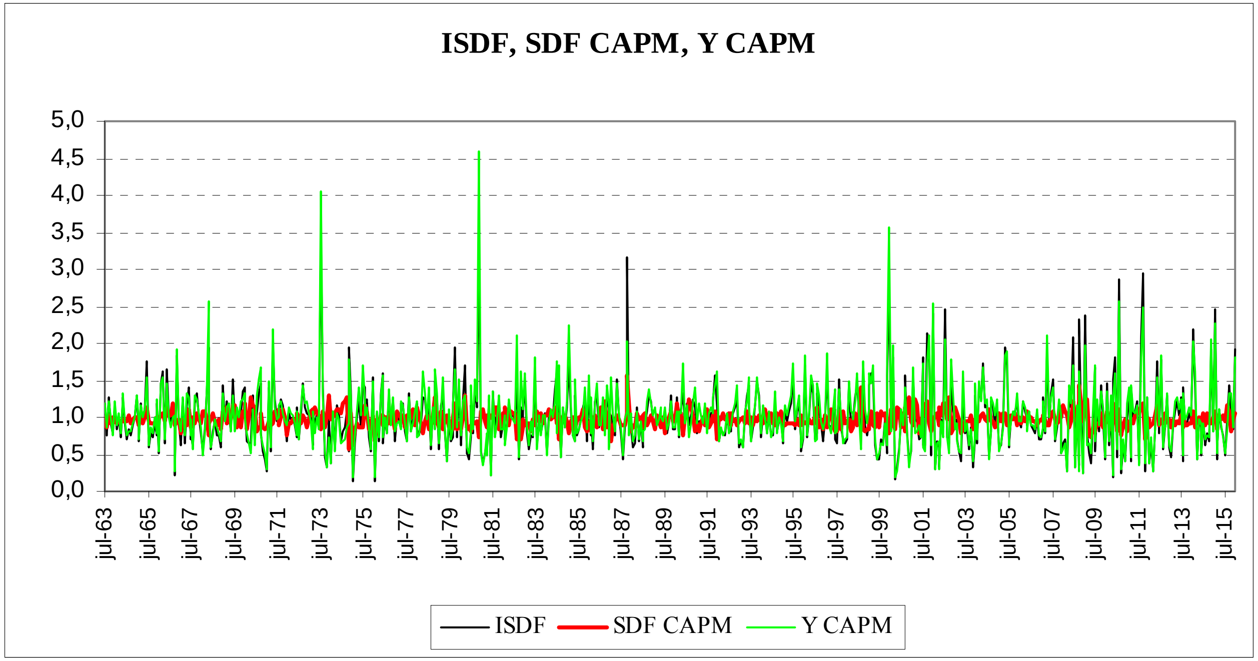


Consumption-Based Asset Pricing Models with Non-Separability





Consumption-Based Asset Pricing Models with Non-Separability



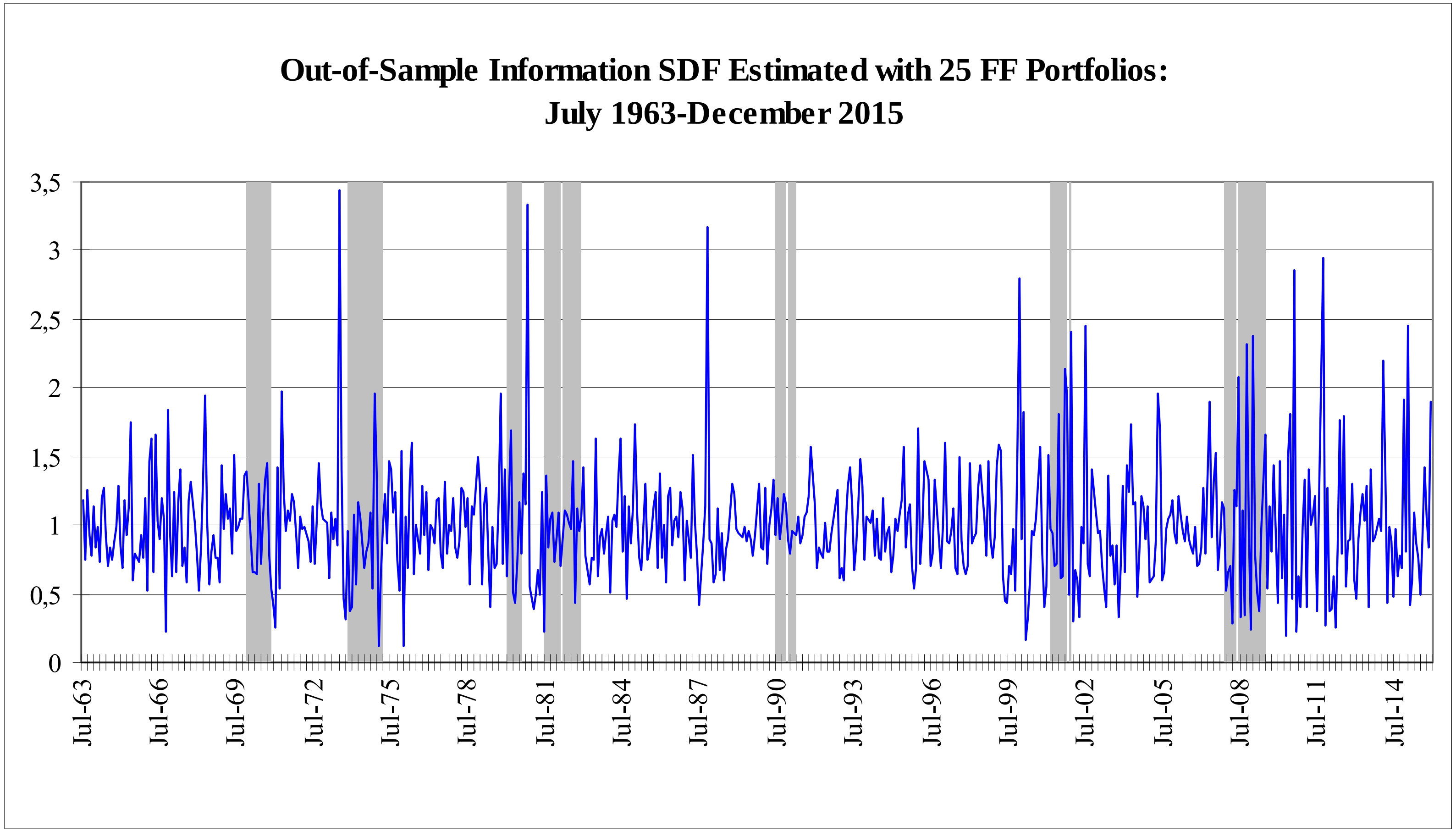
Consumption-Based Asset Pricing Models with Non-Separability

► Well known asset pricing models seem to **miss** a relevant component, which is **highly volatile and especially sensitive to financial (rather than economic) recessions**

► The "residual" Y_{t+1} component captures most of the volatility of the SDF, but very little volatility is generated by the component associated to consumption-based models

How to introduce a recession-related and economically sensible aggregate variable in asset pricing models is the main challenge of Finance!!!

Consumption-Based Asset Pricing Models with Non-Separability



Consumption-Based Asset Pricing Models with Non-Separability

APPENDIX (this is not part of the required material)

The fundamental pricing equation is

$$0 = E^P\left[M_t R_t^e\right] = \int M_t R_t^e dP$$

Let the Radon-Nikodym derivative be

$$\frac{M_t}{\overline{M}} = \frac{dQ}{dP} \quad , \quad \overline{M} = E\left[M_t\right]$$

Then,

$$0 = \int \frac{M_t}{\overline{M}} R_t^e dP = \int R_t^e dQ = E^Q\left[R_t^e\right]$$

Consumption-Based Asset Pricing Models with Non-Separability

We minimize the relative entropy (divergence) of Q relative to P

$$\underset{Q}{arg\ min}\, D(Q \| P) \equiv \underset{Q}{arg\ min} \int \frac{dQ}{dP} ln\left(\frac{dQ}{dP}\right) dP$$

$$subject\ to$$

$$\int R_t^e dQ = 0$$

Given the Radon-Nikodym derivative and the innocuous normalization $\overline{M} = 1$, the optimization above can be written as

$$\underset{M_t}{arg\ min}\, E^P[M_t\, ln\, M_t]$$

$$subject\ to$$

$$E^P\left[M_t R_t^e\right] = 0$$

Consumption-Based Asset Pricing Models with Non-Separability

We replace the expectation with the sample analogue

$$\underset{\{M_t\}_{t=1}^{T}}{\arg\min}\ \frac{1}{T}\sum_{t=1}^{T} M_t \ln M_t$$

subject to

$$\frac{1}{T}\sum_{t=1}^{T} M_t R_t^e = 0$$

Consumption-Based Asset Pricing Models with Non-Separability

It can be shown that the solution can be obtained via the corresponding duality:

$$\hat{M}_t^D \equiv M_t^D\left(\hat{\theta}_T, R_t^e\right) = \frac{e^{\hat{\theta}_T R_t^e}}{\sum_{t=1}^{T} e^{\hat{\theta}_T R_t^e}}, \ \forall t$$

where the estimators of θ is the vector of Lagrange multipliers that solve the unconstrained convex problem

$$\hat{\theta}_T = \arg\min_{\theta} \frac{1}{T} \sum_{t=1}^{T} e^{\theta' R_t^e}$$

which is dual formulation of the entropy minimization problem

Consumption-Based Asset Pricing Models with Non-Separability

Note that given the normalization $\overline{M} = 1$, the solution above produces the demeaned SDF

In order to obtain the monthly Information SDF for an economically reasonable magnitude, we actually employ the following expression

$$\hat{M}_t \equiv M_t\left(\hat{\theta}_T, R_t^e\right) = \frac{T}{R_{ft}} \frac{e^{\hat{\theta}_T R_t^e}}{\sum_{t=1}^{T} e^{\hat{\theta}_T R_t^e}}, \ \forall t$$

where R_{ft} is the gross risk-free rate for each month t